E-16-629

AFOSR–TR–

Final Scientific Report
Grant AFOSR–73–2479

SUMMARY OF RESEARCH ON LINEAR AND NONLINEAR EFFECTS ON VIBRATION OF ELASTIC STRUCTURES

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May 1978

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR -TR-	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SUMMARY OF RESEARCH ON LINEAR AND NONLINEAR EFFECTS ON VIBRATION OF ELASTIC STRUCTURES		5. TYPE OF REPORT & PERIOD COVERED FINAL SCIENTIFIC 1 January 1973 - 31 December 1977 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Lawrence W. Rehfield		8. CONTRACT OR GRANT NUMBER(s) AFOSR-73-2479
9. PERFORMING ORGANIZATION NAME AND ADDRESS Georgia Institute of Technology Aerospace Engineering Atlanta, Georgia 30332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 2307 B1 61102 F
11. CONTROLLING OFFICE NAME AND ADDRESS AF Office of Scientific Research (NA) Building 410 Bolling AFB, D.C. 20332		12. REPORT DATE May 1978 13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release

Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Vibrations. Structural Dynamics. Nonlinear Vibrations. Composites.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report contains a brief summary of research on linear and nonlinear effects in the vibration of elastic structures supported by Grant AFOSR-73-2479 and performed during the period January 1, 1973 to December 31, 1977. The research includes studies of nonlinear vibration of structures, boundary restraint effects in vibration and hygrothermal effects on dynamic behavior of resin matrix composites. A listing of reports and publications that have resulted from this activity is included.

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INTRODUCTION

This report contains a brief summary of research on structural vibration sponsored by the United States Air Force Office of Scientific Research under Grant AFOSR-73-2479. The work has been performed at the School of Aerospace Engineering, Georgia Institute of Technology during the period January 1, 1973 to December 31, 1977. Professor Lawrence W. Rehfield has been the Principal Investigator. This research program is unusual in that a transition from purely theoretical tasks to mostly experimental ones has taken place in the course of the five-year period. This transition occurred in a natural fashion as the need for experimental data in a number of areas became apparent.

The research began with a small scale theoretical investigation of the influence of nonlinear and boundary restraint effects on the vibration of structural elements. The scope of the nonlinear studies broadened to include elements of composite materials. Also, it became apparent that little experimental data on nonlinear structural vibration appeared in the open literature; an experimental task was initiated in late 1974 to provide nonlinear vibration data on beams, therefore. The boundary restraint studies were focused upon the interaction between boundary restraint and element curvature; considerable understanding of this complex issue was gained, but concise, general conclusions proved to be elusive.

Our efforts were redirected in 1976 in response to a pressing problem confronting the aerospace community. The environmental effects of moisture/elevated temperature were identified as a major problem for resin matrix composite structures in late 1975. A shift in emphasis from theory and analysis to laboratory development for basic experiments related to the influence of environmental conditions on dynamic behavior began. The subsequent vibration data produced is the first of its kind.

The scope of the research activity obviously has varied over wide limits during the five-year period. Consequently, a summary cannot be written which unifies the work around a theme more specific that structural vibration. The work described in the following should be viewed, therefore, in the evolutionary context in which it was performed.

RESEARCH ACCOMPLISHMENTS

Introductory Remarks

As indicated in the introduction, the research scope covers a range of activities. For convenience, the work will be placed into three categories. Each category will be treated separately, although there is overlapping among the categories. The first category is nonlinear vibration of structures; this activity is complete in the sense that the entire spectrum from basic concepts to experimental confirmation of theoretical predictions has been explored. Another category is boundary restraint effects; this work, although important, failed to produce a unified, consistent way of conceptualizing general situations. The third category is hygrothermal effects on dynamic behavior of resin matrix composites; only preliminary results are available at this time. This work will continue under a subsequent program.

Nonlinear Vibration of Structures

As efforts continue to produce more highly optimized structures, some effects which have been historically of secondary importance in structural design will tend to assume a primary role. Often these effects are associated with nonlinearity of the structural response. Consequently, much of the research effort has been devoted to exploring the nature and importance of nonlinearity in structural vibrations.

Nonlinear effects in structural dynamics may manifest themselves in subtle ways such as by simple "drift" of natural frequencies with increasing amplitude or by altering the time history of transient response. They also can produce quite dramatic effects that cannot be explained or predicted on the basis of linear theory; among such effects are jump phenomena, dynamic buckling (oil canning) in the presence of prestress or geometric curvature, subharmonic or superharmonic resonances at frequencies far different from the exciting frequency, and bounded limit cycle response. Both the subtle and dramatic effects have been observed in actual structures and have been treated in an ad hoc manner by analysts and engineers. The objective has been to unify and synthesize the treatment of nonlinear effects. In large measure, it has been achieved.

A general theory for large amplitude, free, undamped vibration of linearly elastic structures appears in Reference 1. This pioneering work establishes the framework for all subsequent theoretical effort in nonlinear vibration. The theory is based upon Hamilton's principle and a perturbation procedure and is in much the same spirit as the theory of initial postbuckling behavior due to Koiter.[2,3] It provides information regarding the first order effects of finite displacements upon the frequency and dynamic stresses arising in free, undamped vibration of structures. Solutions to the governing equations are sought as a power series in the amplitude of the linear vibration mode and higher order effects are systematically generated by successive perturbation equations. The first applications were to homogeneous uniform beams and rectangular plates.[1]

The general theory has been applied also to the free vibration of shallow arches as an initial study of the effect of curvature on nonlinear vibration behavior.[4] It has been learned that as the rise of the arch increases, the free vibration behavior in the fundamental mode first exhibits a softening trend due to curvature. This trend is reversed as the rise parameter exceeds a critical value, and a neutral limit is ultimately approached for large values of rise.

The theory subsequently was extended to include forced vibrations of linearly elastic structures in References 5 and 6. It has been assumed that the structure is excited by a distrubing force which is a harmonic function of time and is spatially distributed so as to excite only a single mode. General results are presented [5,6] along with applications to the vibration of beams and rectangular plates. Other solution methods have been compared with the general theory for the beam and the results are found to be in complete agreement.

A further extension of the theory was accomplished to include the effects of nonlinear elastic constitutive relations in both free and forced vibration.[7] As a first approximation, it was assumed that tensile and compressive behavior of the materials are similar and that the stress-strain relations can be adequately represented as polynomial functions. The theory was first applied to the undamped free vibration of a simply supported beam with immovable ends. Numerical results have been presented for several materials, for beams of varying slenderness, and for fundamental and higher vibration modes to illustrate the nonlinear material effects (NLME's). The results obtained indicate that NLME's are extremely important and dominantly influence the behavior for higher modes and short beams. They also suggest that the second order theory, in which the frequency is altered by NLME's but the stress distribution is the same as for Hookean elastic materials, is fully adequate for characterizing the elastic vibration of structures made of conventional homogeneous materials.

In Reference 8, an approximate method for analyzing nonlinear vibration of elastic structures is presented. It is based upon the general theory[1,6,7] and permits the engineer to obtain a rapid assessment of nonlinear effects in the fundamental mode of vibration. It represents a logical extension of Rayleigh's method to nonlinear problems. Applications to a number of uniform beam problems[8] have been made to illustrate the use of the method. The results obtained also serve to further emphasize the importance of NLME's and boundary restraint effects in structural vibrations.

In early 1976, many aspects of the moisture/elevated temperature or hygrothermal problem of resin matrix composites were not adequately defined. The combined difficulties posed by moisture/temperature simulation and the variety of composite lay-ups and material systems of interest resulted in an incomplete, fragmented data base. Test methods had been developed on an ad hoc basis, so comparison and correlation of data was often not rationally possible. Consequently, the scope and magnitude of the effects still lacked sufficient definition. One effect that was documented, however, is the degradation of compressive stiffness. A pioneering paper[9] on the effects of nonlinear material behavior on the vibration of resin matrix composites was presented at the 17th AIAA/ASME Structures, Structural Dynamics and Materials Conference. In this paper, an estimate of the effect of compressive stiffness degradation on the vibration of [0] laminated beams is provided. It is concluded that this effect will not produce important consequences for fiber-dominated flexural vibrations. It is further concluded that only matrix-dominated flexure, such as that experienced during lateral vibration of angle ply panels, can potentially exhibit important nonlinear material effects.

It was recognized early in this nonlinear vibration program that experimental work is scarce and greatly needed. There are difficulties, however, in performing experiments in nonlinear parametric ranges. For example, conventional electromagnetic shakers, which are designed for resonance testing, are both stroke and force limited; consequently, it is not a straight-forward matter to excite beam specimens to amplitudes that are sufficiently large to produce nonlinear effects over the required range of frequencies. A number of exploratory experimental setups were tried and found inadequate. The first successful experimental approach is described in Reference 10. The unique feature about this approach is the use of longitudinal rather than lateral excitation to produce large amplitude motion in beams. The phenomenon produced, parametric excitation, is intrinsically nonlinear in character. Consequently, it is not possible to determine linear theory, small amplitude response with this experimental set up. Since linear as well as nonlinear characteristics of the vibration are needed for a complete description, an alternative is needed.

A second independent experimental effort was under taken in early 1977 under the direction of Dr. Giora Maymon. This approach is based upon the general theory [1,6,7] developed previously. A useful indicator of the magnitude and nature (hardening or softening) of nonlinear stiffness-related effects is the frequency-amplitude relation for undamped, free vibration in a given vibration mode. For most structures, this relation is of the form

$$\frac{\omega_r^2}{\omega_{or}^2} = 1 + B_r \, (\xi_r)^2 \tag{1}$$

ω_r is the response frequency, ω_{or} is the natural frequency according to linear vibration theory, and ξ_r is the normalized amplitude of the r-th vibration mode. B_r is the coefficient of nonlinearity for the r-th mode; its sign and magnitude characterize the nonlinear effects.

Attention is directed to vibration of elastic structures that respond by simple jump behavior. A typical nonlinear response curve is shown in Figure 1. While the response curve for a linear system is almost symmetric about the undamped natural frequency, this curve is distorted and "overhangs" (to the right (hardening behavior) in the case depicted in Figure 1). Also, multivalued amplitudes exist for the same value of frequency. Jumps occur if the frequency is varied; a "jump down" occurs as the frequency is increased. A "jump up" takes place as the frequency is decreased. The "backbone curve" indicated in Figure 1 corresponds to undamped, free vibration. It is mathematically described by Equation (1). If the exciting force magnitude is varied, a family of response curves similar to the one shown are produced. Each such curve will have a response point corresponding to where the jump down occurs. If the damping of the system is small and independent of amplitude, these points corresponding to "jump down" will, for practical experimental purposes, also correspond to the maximum amplitude at a given level of excitation and to the intercept of the response curve with the backbone curve. Consequently, the maximum amplitudes achieved in a test are interpreted as being on the backbone curve and corresponding to the "jump down" frequency. This is the essential basis for the experimental method.

In order to evaluate this method of experimentally determining nonlinear effects, a simple aluminum clamped-clamped beam with immovable ends was chosen as the structural system. The beam is excited laterally with two small exciters at points close to the clamped ends; these excitation points were selected so that only a small force is needed to excite rather large amplitudes. The exciters are attached to the beam specimen through small universal joints to avoid any side loads during flexural rotations. Miniature piezoelectric load cells sensed excitation levels. Accelerations were measured by a small accelerometer and frequency by a digital counter. The experimental setup is shown schematically in Figure 2.

A typical experimentally determined response curve is shown in Figure 3. The 'open' symbols correspond to a scan with increasing frequency, while the corresponding 'solid' symbols refer to a scan with decreasing frequency. The jumps are clearly and easily discernible. The primary results of interest are shown in Figure 4. If Equation (1) describes the structural system, a plot of frequency squared vs. amplitude squared will be linear. Furthermore, the intercept with the frequency squared axis provides the linear theory natural frequency and the slope gives the coefficient of nonlinearity. Figure 4 is self explanatory. The natural frequency has been determined also in a separate small amplitude transient test; the two experiments are in agreement.

The coefficient of nonlinearity determined from the data presented in Figure 4 is five percent larger than the theoretical value given in Reference 11. Consequently, the data confirms the theory and the physical approximations upon which the experimental method is based. These results have been presented as Reference 12.

In addition to the above experimental approach, three methods for determining the damping coefficient for a nonlinear vibrating structural system have been found.[13] All three are based upon the fact that the frequencies and amplitudes of the motion that correspond to jumps are well-defined, easily determined points. The methods, therefore, have no counterparts for linear systems.

It is seen that the nonlinear vibration work reached a state of completeness and maturity for simple structural systems with Reference 12. Issues such as modal coupling or interaction remain as topics for future research.

<u>Boundary Restraint Effects</u>

The engineer in seeking to avoid or solve a vibration problem most often will alter structural stiffness to achieve the desired results. Stiffness of an element is changed by

(1) selecting a new material

(2) selecting another gauge of the same material

(3) altering and/or introducing a favorable prestress field in the element

(4) modifying element boundary restraint

(5) altering and/or introducing favorable geometric curvature

(6) redesigning the element

Several of these actions may be taken in a given situation depending upon the stage in the design process that the problem is discovered, governing economic factors and the particular performance requirements of the system. Since unwanted resonance and fatigue are usually the concern, increased stiffness is normally prescribed.

Stringent weight requirements for flight vehicle airframe structure dictate that optimal solutions be found to vibration problems whenever possible. For this reason, items (4) and (5) and their interaction were investigated using linear theory. The addition of a small amount of fastener weight, for example, to increase boundary restraint may effectively result in a large increase in natural frequencies for structural elements with geometric curvature. Evidence supporting this contention for curved panels is provided by results published in Reference 14.

The analysis of Reference 14 demonstrates beyond doubt the importance of boundary restraint and a great sensitivity of the response to seemingly minor changes in the details of the restraint. In order to harness the potential benefits for achieving desirable results in actual structures, additional basic knowledge and an understanding of the governing mechanisms must be acquired. A step in this direction was taken in 1974 by the thorough study of the vibration of ring segments reported in Reference 15.

Circular ring segments, in addition to being of practical interest in connection with floating ring stiffened shell design, are the simplest structural elements upon which to study the influence of boundary restraint and curvature. Results are presented in Reference 15 for vibration modes which are symmetric and anti-symmetric about the segment semi-span, for limiting cases of bending and extensional restraint and also for finite extensional elastic end springs. Definite patterns in the vibration behavior have been found. It was learned that the phenomena of frequency reversal among anti-symmetric and symmetric vibration modes as curvature is increased (the symmetric mode's corresponding frequency increases with curvature to the point when it becomes greater than an anti-symmetric mode's frequency, producing an exchange or reversal of position of the modes with regard to frequency) very nearly corresponds to a maximum value of extensional strain energy in the symmetric mode. The correspondence is not an exact one, but so nearly so that it tends to identify the ratio of extensional to total strain energy as a measure of merit in evaluating frequency reversal phenomena and for evaluating the beneficial, stiffening effect of geometric curvature.

The above hypothesis was tested by returning to a further study of curved panels to see if the ring segment observations have wider applicability. Unfortunately, they do not. Although the extensional strain energy ratio is a significant parameter, all efforts to draw general conclusions beyond those reported in References 14 and 15 were unsuccessful. The investigation, therefore, was terminated at this point.

Hygrothermal Effects on the Dynamic Behavior of Resin Matrix Composites

Epoxy resins used as matrix materials in advanced structural composites experience degradation of mechanical properties due to moisture absorption and elevated temperature environments. It is this fact that caused a redirection of the research effort in 1976. Considerable recent and current work has been directed toward documenting the extent of this degradation for simple layup configurations and states of stress. Primary emphasis has been placed upon static performance. An exception is the work pursued under this grant.

As indicated previously, a preliminary estimate of the impact of hygrothermal effects on vibration of beams is given in Reference 9. This has been followed by the pioneering experimental study by Maymon, Briley and Rehfield[16] of the influence of moisture/temperature effects on dynamic structural behavior of epoxy matrix composite beams in flexure.

This study showed the contrast between two limiting reference conditions -- dry at room temperature ($77^{o}F$) and moisture saturation at $200^{o}F$. Changes in fundamental natural frequency, an overall, practical measure of stiffness, and damping were determined experimentally. The objective was to establish a data base to inspire confidence in the use of graphite/epoxy composites in hostile, dynamic environments.

In addition to providing the first data on hygrothermal effects on damping, there is another aspect of this work that warrants special mention. An effort was made to approximate the environmental conditions <u>during the test</u>. The vibration testing technique was dictated by the need for simplicity, repeatability and preservation of the environmental conditions. In order to preserve the environmental state to the maximum extent, the excitation technique selected for vibration testing was a transient one. The advantage obtained is a short duration test. The beams are mounted in cantilever fashion in a fixture with prescribed tip deflection. This setup is shown in Figure 5. The tip is suddenly released producing transient response of the specimen that is dominated by the fundamental mode of vibraton. Response is sensed by an accelerometer mounted near the specimen tip. The data are processed by means of a minicomputer-based Fourier analyzer which incorporates an analog to digital converter. The digitized accelerometer output is printed out, and the fundamental frequency and corresponding damping coefficient determined by simple analyses. Frequency is determined simply by counting the number of peak values contained in a suitably chosen time interval. Values for the damping coefficient are obtained by the logarithmic decay method.

Three distinct composite layups have been considered. The layups are $[0]$, $[\pm 45]$, and $[0_2, +45_2, 90_1, -45_1]$s. They were each 12-plies thick and symmetric and will be referred to as type A, B, and C, respectively.

The material is graphite/epoxy T300/5208. Types A, and B are limiting types of fundamental interest, while C is representative of practical vehicle skin panels. The A-specimens respond in a fiber controlled mode of vibration, while the B-specimens respond in a matrix controlled mode. The results indicate that stiffness is reversibly lowered in matrix controlled modes of deformation and that damping characteristics are altered substantially in both fiber and matrix controlled modes due to environmental conditioning. The C-specimens exhibit stiffness characteristics that appear to be fiber controlled (unchanged); the damping, however, is similar in nature to matrix controlled behavior. The hot, wet environment increases damping for A-type specimens, while it decreases damping markedly for B- and C-type ones. The latter is consistent with simple logic based upon decreasing effective internal viscosity of the epoxy matrix material with increasing moisture and temperature. The former could be related to increased relative motion between fibers and matrix at the interface.

The initial phase of the study of hygrothermal effects on dynamic behavior appears in Reference 16. This work will be continued and broadened in scope under a new contract.

CONCLUDING REMARKS

The research work summarized in the foregoing pages represents a long journey in the minds of the participants. Basic theoretical issues as well as practical experimentation have been encountered at different points along the way. As mentioned in the introduction, this is an unusual research program because of this variation in scope. Mr. William J. Walker of AFOSR provided the opportunity for response to the problem posed by hygrothermal effects in resin matrix composites. Such a challenging opportunity is rare; it is gratefully acknowledged.

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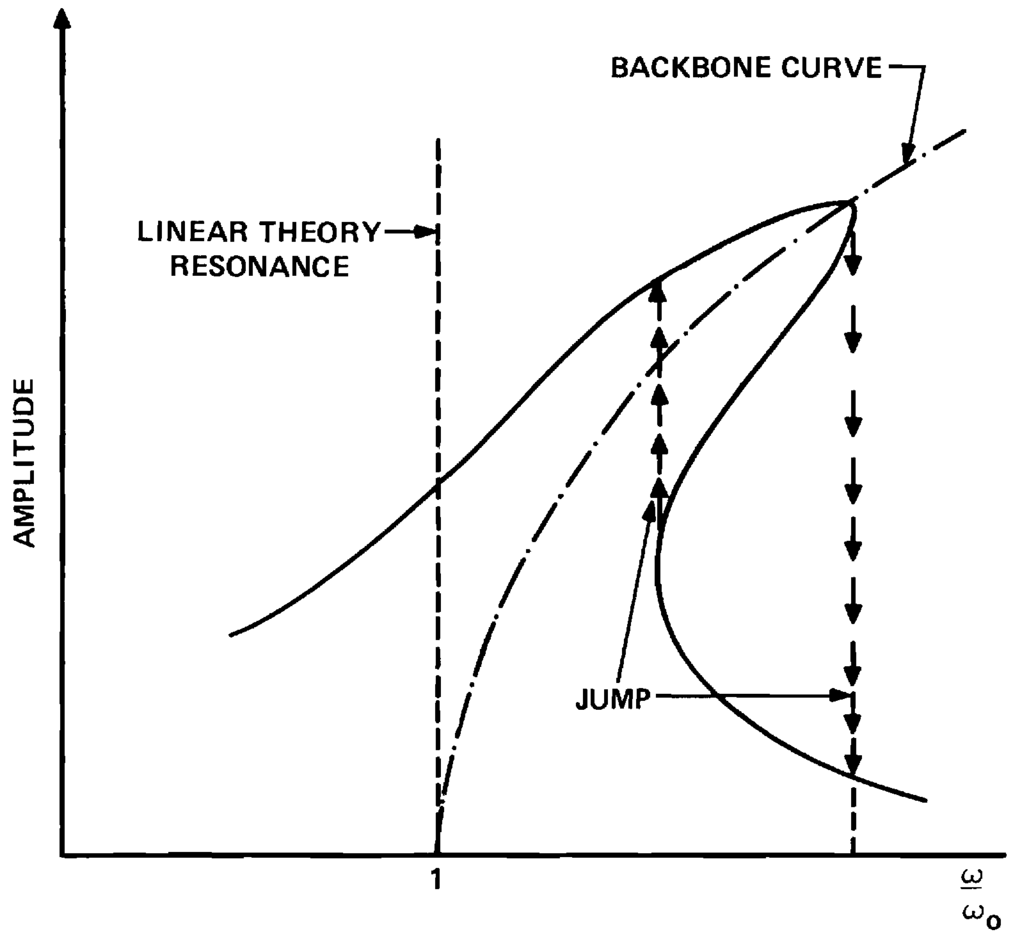

Figure 1. Typical Nonlinear Forced Response Curve for a Hardening Elastic System

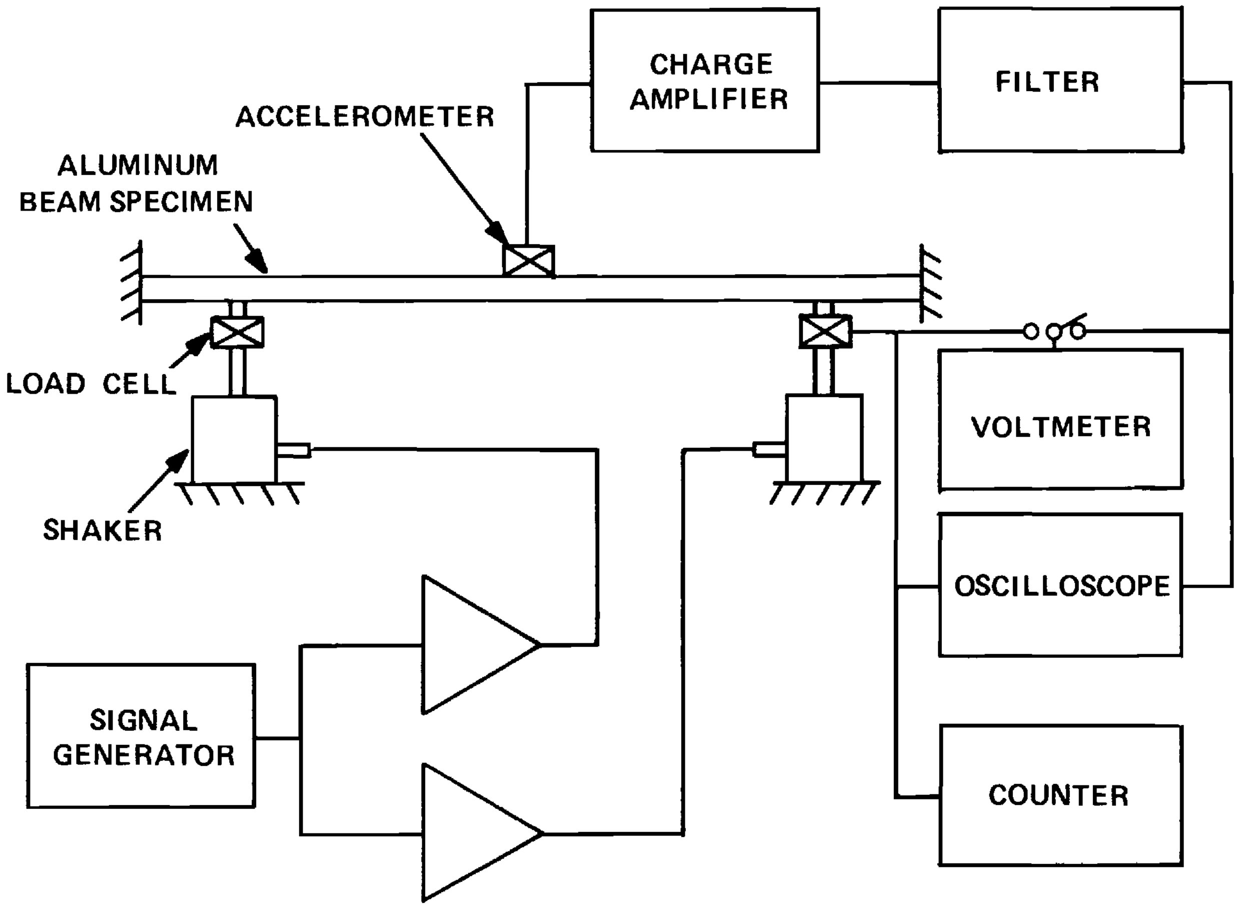

Figure 2. Schematic of Experimental Setup for Nonlinear Vibration Study

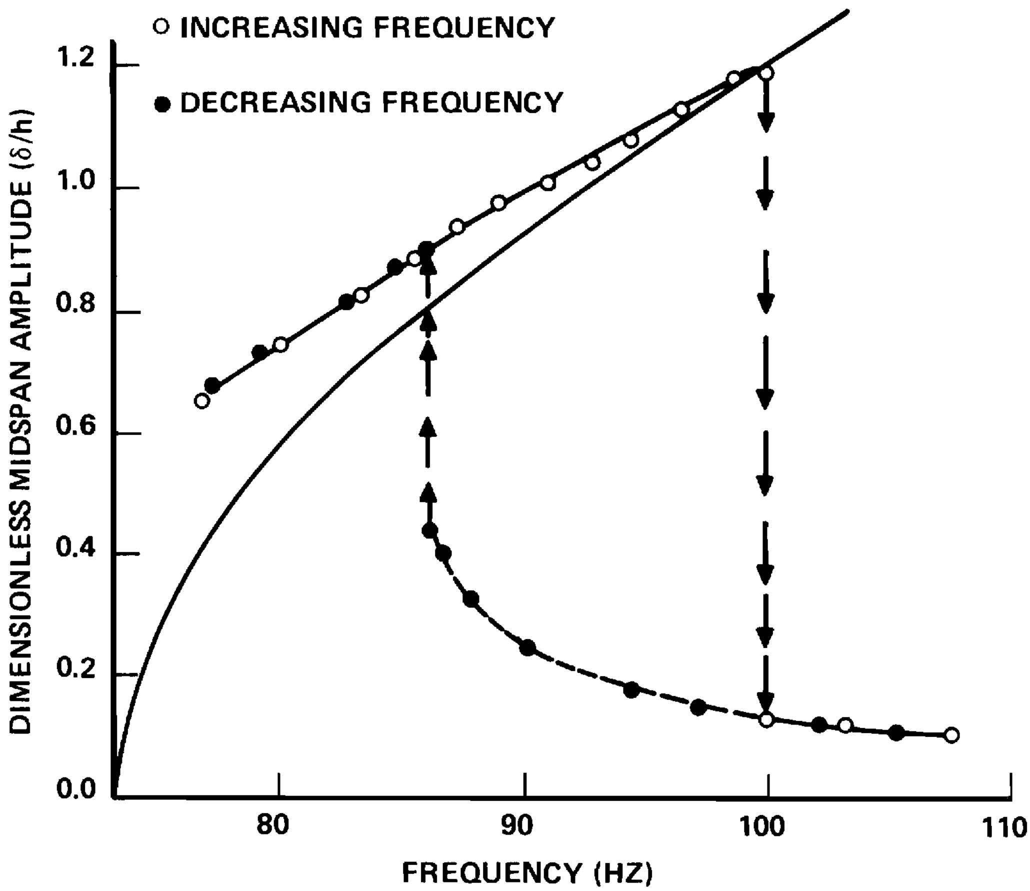

Figure 3. Typical Experimentally Determined Nonlinear Forced Response Data

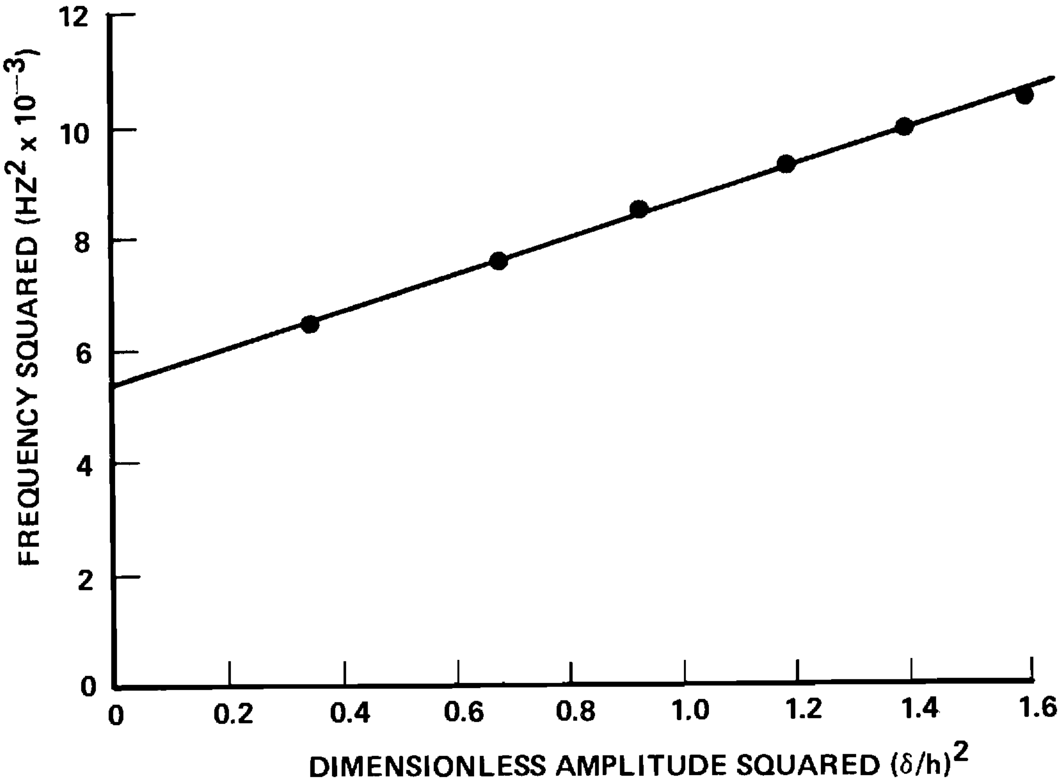

Figure 4. Nonlinear Stiffness Plot
(Frequency Squared vs. Amplitude Squared)

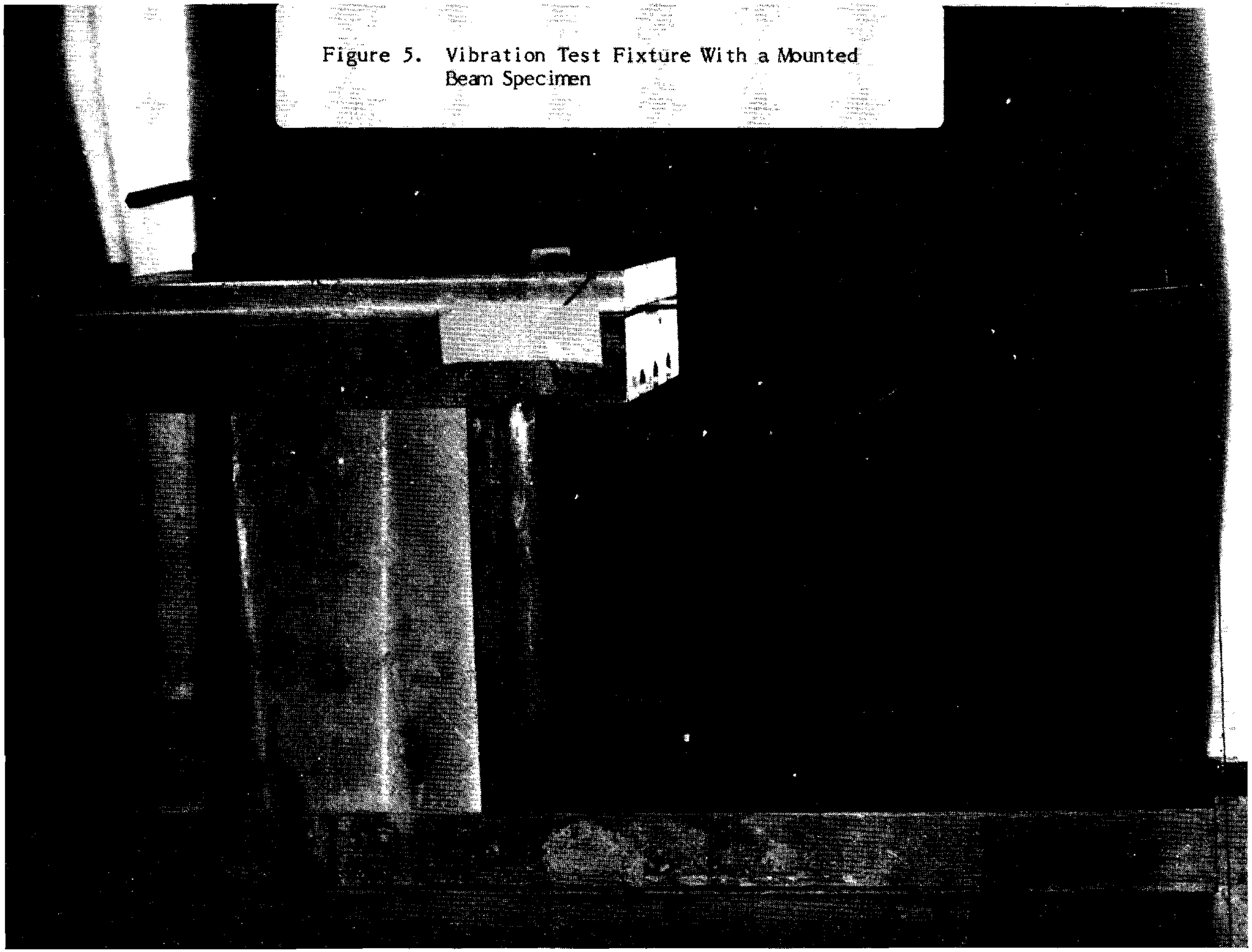

Figure 5. Vibration Test Fixture With a Mounted Beam Specimen

Reprinted from AIAA JOURNAL, Vol. 12, No. 1, January 1974, pp. 91-93

Nonlinear Flexural Oscillations of Shallow Arches

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Introduction

NONLINEAR oscillations of elastic systems with curvature have received little attention in the literature. The systems that have received the most attention are circular rings[1,2] and circular cylindrical shells.[3-7] The effect of curvature on dynamic behavior must be more fully understood in order to effectively deal with shell structures. In an effort to explore this effect, nonlinear flexural oscillations of shallow arches are studied in the present Note.

The analysis is conducted within the scope of a general approach to nonlinear free vibration of elastic structures reported in detail elsewhere.[8] The approach is briefly outlined herein for completeness. It is analogous to the theory of initial postbuckling behavior due to Koiter[9,10] and provides information regarding the first-order effects of finite displacements.

Outline of the General Theory

To facilitate a concise presentation, the functional notation used by Budiansky[11] will be employed. The motion of the structure produces generalized displacement $\mathbf{u}$, strain γ and stress $\boldsymbol{\sigma}$. The motion is assumed to be periodic such that

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{u}[\mathbf{r}, t+(2\pi/\omega)] \tag{1}$$

where $\mathbf{r}$ is the position vector to an arbitrary point in the structure and ω is the circular frequency.

The system dynamics is established by Hamilton's principle, which is symbolically written

$$\int_0^{2\pi/\omega} \left\{\delta\left[\frac{1}{2}M\left(\frac{\partial \mathbf{u}}{\partial t}\right)\cdot\frac{\partial \mathbf{u}}{\partial t}\right] - \boldsymbol{\sigma}\cdot\delta\gamma\right\} dt = 0 \tag{2}$$

The "dot" operation signifies the appropriate inner multiplication of variables and integration of the result over the entire structure. The generalized mass operator M is assumed to be homogeneous and linear with the property that

$$M(\mathbf{u})\cdot\mathbf{v} = M(\mathbf{v})\cdot\mathbf{u} \tag{3}$$

for all $\mathbf{u}$ and $\mathbf{v}$. Since only periodic motion is to be considered, the limits on the integral over time correspond to a single period of the motion. If a new time variable $\tau = \omega t$ is introduced in Eq. (2) and an integration by parts is performed, the boundary terms vanish leaving

$$\int_0^{2\pi} [\omega^2 M(\ddot{\mathbf{u}})\cdot\delta\mathbf{u} + \boldsymbol{\sigma}\cdot\delta\gamma]\, d\tau = 0 \tag{4}$$

In the above, the notation $(\dot{\ }) = \partial(\ )/\partial\tau$ has been used. $\delta\mathbf{u}$ is any virtual displacement that is consistent with all the kinematic boundary conditions imposed on the structure.

Nonlinear geometric effects enter through the strain-displacement relation, which may be symbolically written

$$\gamma = \mathbf{e} + \tfrac{1}{2}L_2(\mathbf{u}) \tag{5}$$

Received September 27, 1972; revision received August 17, 1973. This work was supported by the U.S. Air Force Office of Scientific Research under Grant AFOSR-73-2479.

Index category: Dynamic Structural Analysis.

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where $\mathbf{e}$ is the linearized strain measure and L_2 is a homogeneous quadratic functional. In addition, the homogeneous bilinear functional L_{11} is defined by the following equation

$$L_2(\mathbf{u}+\mathbf{v}) = L_2(\mathbf{u}) + 2L_{11}(\mathbf{u}, \mathbf{v}) + L_2(\mathbf{v}) \tag{6}$$

It follows that $L_{11}(\mathbf{u}, \mathbf{v}) = L_{11}(\mathbf{v}, \mathbf{u})$ and $L_{11}(\mathbf{u}, \mathbf{u}) = L_2(\mathbf{u})$. Consequently

$$\delta\gamma = \delta\mathbf{e} + L_{11}(\mathbf{u}, \delta\mathbf{u}) \tag{7}$$

The stress-strain relation is taken to be

$$\boldsymbol{\sigma} = H(\gamma) \tag{8}$$

H is a homogeneous linear functional. The following reciprocity relation

$$\boldsymbol{\sigma}^{(1)}\cdot\gamma^{(2)} = \boldsymbol{\sigma}^{(2)}\cdot\gamma^{(1)} \tag{9}$$

will be assumed also; "1" and "2" are any arbitrary states of stress and strain.

The response of the structure is found by setting

$$\begin{aligned}\mathbf{u} &= \xi\mathbf{u}_1 + \xi^2\mathbf{u}_2 + \cdots \\ \gamma &= \xi\mathbf{e}_1 + \xi^2[\mathbf{e}_2 + \tfrac{1}{2}L_2(\mathbf{u}_1)] + \cdots \\ \boldsymbol{\sigma} &= \xi\boldsymbol{\sigma}_1 + \xi^2\boldsymbol{\sigma}_2 + \cdots\end{aligned} \tag{10}$$

ξ is an amplitude parameter associated with the linear vibration mode $\mathbf{u}_1$ which has natural frequency ω_0. If Eq. (10) is substituted into Eq. (4), we obtain

$$\begin{aligned}\int_0^{2\pi} \{&\xi[\omega^2 M(\ddot{\mathbf{u}}_1)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_1\cdot\delta\mathbf{e}] + \\ &\xi^2[\omega^2 M(\ddot{\mathbf{u}}_2)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_2\cdot\delta\mathbf{e} + \boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1, \delta\mathbf{u})] + \\ &\xi^3[\omega^2 M(\ddot{\mathbf{u}}_3)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_3\cdot\delta\mathbf{e} + \boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_2, \delta\mathbf{u}) + \\ &\boldsymbol{\sigma}_2\cdot L_{11}(\mathbf{u}_1, \delta\mathbf{u})] + \cdots\}\, d\tau = 0\end{aligned} \tag{11}$$

This equation is satisfied progressively by requiring the coefficient of each power of ξ to vanish independently. The vanishing of the linear term in ξ yields the linearized equation of free vibration

$$\omega_0^2 M(\ddot{\mathbf{u}}_1)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_1\cdot\delta\mathbf{e} = 0 \tag{12}$$

If we set $\delta\mathbf{u} = \mathbf{u}_1$ and $\delta\mathbf{e} = \mathbf{e}_1$ in the aforementioned Eq. (12) and integrate the result, we obtain the following expression for ω_0^2

$$\omega_0^2 = \frac{-\int_0^{2\pi} \boldsymbol{\sigma}_1\cdot\mathbf{e}_1\, d\tau}{\int_0^{2\pi} M(\ddot{\mathbf{u}}_1)\cdot\mathbf{u}_1\, d\tau} = \frac{\int_0^{2\pi} \boldsymbol{\sigma}_1\cdot\mathbf{e}_1\, d\tau}{\int_0^{2\pi} M(\dot{\mathbf{u}}_1)\cdot\dot{\mathbf{u}}_1\, d\tau} \tag{13}$$

We assume at this point that a single mode $\mathbf{u}_1$ is associated with the natural frequency ω_0. In order to make the expansions unique, the displacement increments $\mathbf{u}_2, \mathbf{u}_3, \ldots$ are orthogonalized with respect to $\mathbf{u}_1$ in the sense that

$$\int_0^{2\pi} M(\dot{\mathbf{u}}_1)\cdot\dot{\mathbf{u}}_k\, d\tau = \int_0^{2\pi} M(\ddot{\mathbf{u}}_1)\cdot\mathbf{u}_k\, d\tau = 0 \qquad (k \neq 1) \tag{14}$$

This relation, together with Eq. (12), also implies that

$$\int_0^{2\pi} \boldsymbol{\sigma}_1\cdot\mathbf{e}_k\, d\tau = 0 \qquad (k \neq 1) \tag{15}$$

which, by virtue of the reciprocity relation (9) implies further that

$$\int_0^{2\pi} H(\mathbf{e}_k)\cdot\mathbf{e}_1\, d\tau = 0 \qquad (k \neq 1) \tag{16}$$

If we now set $\delta\mathbf{u} = \mathbf{u}_1$, $\delta\mathbf{e} = \mathbf{e}_1$ in Eq. (11) and introduce Eq. (13), we obtain

$$\int_0^{2\pi} \{\xi[1-(\omega^2/\omega_0^2)]\boldsymbol{\sigma}_1\cdot\mathbf{e}_1+\xi^2[\boldsymbol{\sigma}_2\cdot\mathbf{e}_1+\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)]+$$
$$\xi^3[\boldsymbol{\sigma}_3\cdot\mathbf{e}_1+\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_2)+\boldsymbol{\sigma}_2\, L_{11}(\mathbf{u}_1,\mathbf{u}_1)]+\cdots\}\,d\tau=0$$

However, the reciprocity relation (9) and the orthogonality relation (14) permit the above to be written

$$\int_0^{2\pi} \{\xi[1-(\omega^2/\omega_0^2)]\boldsymbol{\sigma}_1\cdot\mathbf{e}_1+\xi^2[\tfrac{3}{2}\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)]+$$
$$\xi^2[2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_2)+\boldsymbol{\sigma}_2\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)]+\cdots\}\,d\tau=0 \tag{17}$$

Consequently, we have the asymptotic relation

$$\omega^2/\omega_0^2 = 1+A\xi+B\xi^2+\cdots \tag{18}$$

where

$$A=\frac{\int_0^{2\pi}\frac{3}{2}\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)\,d\tau}{\int_0^{2\pi}\boldsymbol{\sigma}_1\cdot\mathbf{e}_1\,d\tau} = \frac{\int_0^{2\pi}\frac{3}{2}\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)\,d\tau}{\omega_0^2\int_0^{2\pi}M\dot{\mathbf{u}}_1)\cdot\dot{\mathbf{u}}_1\,d\tau} \tag{19}$$

and

$$B=\frac{\int_0^{2\pi}[2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_2)+\boldsymbol{\sigma}_2\cdot L(\mathbf{u}_1,\mathbf{u}_1)]\,d\tau}{\omega_0^2\int_0^{2\pi}M(\dot{\mathbf{u}}_1)\cdot\dot{\mathbf{u}}_1\,d\tau} \tag{20}$$

If A is nonzero, the structure can exhibit a softening characteristic with the frequency decreasing for finite amplitudes for $A\xi$ negative. If A is zero, a negative value of B corresponds to softening (decreasing frequency) and a positive, nonzero value corresponds to hardening (increasing frequency). If both A and B are zero, higher order terms must be investigated to discover the nature of finite amplitude effects.

In the evaluation of B a solution for $\mathbf{u}_2$ and $\boldsymbol{\sigma}_2$ is required. It may be found by substituting Eq. (18) in Eq. (11) and setting the coefficient of ξ^2 to zero. The variational equation of motion is therefore

$$\omega_0^2 M(\ddot{\mathbf{u}}_2)\cdot\delta\mathbf{u}+\boldsymbol{\sigma}_2\cdot\delta\mathbf{e}+\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\delta\mathbf{u})=0 \tag{21}$$

and

$$\boldsymbol{\gamma}_2=\mathbf{e}_2+\tfrac{1}{2}L(\mathbf{u}_1,\mathbf{u}_1)$$
$$\boldsymbol{\sigma}_2=H(\boldsymbol{\gamma}_2)$$

$\delta\mathbf{u}$ is orthogonal to $\mathbf{u}_1$ in the sense of Eq. (14).

Equations for the Shallow Arch

A uniform, shallow arch with rise $Z_0(x)$ and projected length L is shown in Fig. 1 along with notation and a sign convention. U and W are displacement components in X and Z directions, respectively. If the effect of longitudinal inertia on the motion is neglected, the axial force N is a function of time only and is given by

$$N=EA[U_{,x}+Z_{0,x}W_{,x}+\tfrac{1}{2}(W_{,x})^2] \tag{22}$$

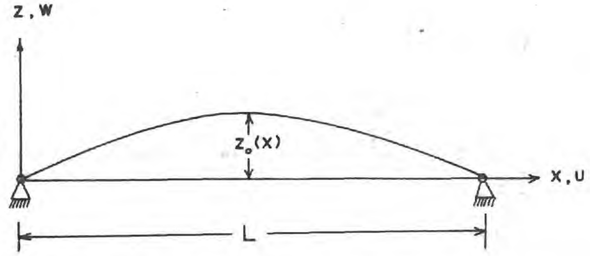

Fig. 1 Shallow arch with notation and sign convention.

E is Young's modulus and A is the cross-sectional area. If the ends of the arch are assumed to be supported such that relative motion between them is prevented, then

$$U(L)-U(0)=\int_0^L U_{,x}\,dX=0 \tag{23}$$

This requirement leads to the following expression for the axial force

$$N=\frac{EA}{L}\int_0^L[Z_{0,x}W_{,x}+\tfrac{1}{2}(W_{,x})^2]\,dX \tag{24}$$

The transverse equation of motion is

$$mW_{,tt}+EIW_{,xxxx}-N(W_{,xx}+Z_{0,xx})=0 \tag{25}$$

where m is the mass per unit length of the beam and EI is its bending stiffness.

We introduce the rise parameter H defined such that

$$Z_0=H\tilde{Z}_0 \tag{26}$$

The maximum value of $\tilde{Z}_0$ is unity. Also, we introduce the following quantities and parameters

$$n=L^2N/\pi^2EI \qquad x=\pi X/L$$
$$w=W/(\pi)^{1/2}\rho \qquad 1/\omega_1^2=(L)^4m/\pi^4EI \qquad r=H/(\pi)^{1/2}\rho \tag{27}$$

$\rho=(I)^{1/2}/A$ is the radius of gyration of the beam cross section and ω_1 is the frequency of the fundamental mode of the corresponding straight beam. If the previous definitions are utilized, then the governing equations may be written

$$n=\int_0^\pi[r\tilde{Z}_{0,x}w_{,x}+\tfrac{1}{2}(w_{,x})^2]\,dx \tag{28}$$

$$[1/(\omega_1)^2]w_{,tt}+w_{,xxxx}-n(w_{,xx}+r\tilde{Z}_{0,xx})=0 \tag{29}$$

Solution Corresponding to the Fundamental Mode

The shape of the arch is taken as a half-sine function

$$\tilde{Z}_0=\sin x \tag{30}$$

This arch model has been used extensively in both static and dynamic studies[12–16] and leads to considerable analytical simplifications.

We take the solutions to Eqs. (28) and (29) as expansions of the form

$$n=\xi n_1+\xi^2 n_2+\cdots$$
$$w=\xi w_1+\xi^2 w_2+\cdots \tag{31}$$

The fundamental mode of the arch is the most interesting and it corresponds to the solution (for simply supported ends)

$$w_1=\sin x\cos\omega t \tag{32}$$

If we set $\tau=\omega t$ and $\Omega=(\omega)^2/(\omega_1)^2$, the first linear approximation resulting from Eq. (32) is

$$n_1=r\int_0^\pi\tilde{Z}_{0,x}w_{1,x}\,dx=(\pi r/2)\cos\tau \tag{33}$$

$$\Omega_0=(\omega_0)^2/(\omega_1)^2=1+(\pi r^2/2) \tag{34}$$

The linearized theory, therefore, predicts a frequency increase with the rise parameter r.

The second-order equations are

$$n_2=r\int_0^\pi\cos x w_{2,x}\,dx+\frac{1}{2}\int_0^\pi(w_{1,x})^2\,dx \tag{35}$$

$$\Omega_0\ddot{w}_2+w_{2,xxxx}+r\sin x n_2-n_1w_{1,xx}=0 \tag{36}$$

where $(\dot{\ })=\partial(\ )/\partial\tau$.

These equations admit a solution of the form

$$w_2=\phi(\tau)\sin x \tag{37}$$

The function ϕ which satisfies Eqs. (35) and (36) and the orthogonality relation (14) is

$$\phi=(\pi r/8\Omega_0)(\cos 2\tau-3) \tag{38}$$

Thus

$$n_2=(\pi^2r^2/16\Omega_0)(\cos 2\tau-3)+(\pi/8)(1+\cos 2\tau) \tag{39}$$

The parameter A is given by

$$A = \frac{\frac{3}{2}\int_0^{2\pi}\int_0^{\pi} n_1(w_{1,x})^2\,dx\,d\tau}{\Omega_0 \int_0^{2\pi}\int_0^{\pi} (\dot{w}_1)^2\,dx\,d\tau} = 0 \tag{40}$$

and B is determined to be

$$B = \frac{2\int_0^{2\pi}\int_0^{\pi} n_1 w_{1,x} w_{2,x}\,dx\,d\tau + \int_0^{2\pi}\int_0^{\pi} n_2(w_{1,x})^2\,dx\,d\tau}{\Omega_0 \int_0^{2\pi}\int_0^{\pi} (\dot{w}_1)^2\,dx\,d\tau} \tag{41}$$

$$= (3\pi/16\Omega_0)[1-(5\pi r^2/2\Omega_0)]$$

Consequently

$$\Omega/\Omega_0 = 1+(3\pi/16\Omega_0)[1-(5\pi r^2/2\Omega_0)]\xi^2+\cdots \tag{42}$$

Initially as r increases, the behavior trend is one of softening (decreasing frequency). This persists until

$$r^2 = 3/\pi \qquad B_{\min} = -3\pi/20 \tag{43}$$

Thereafter a reversed hardening trend appears that approaches the neutral limit $B = 0$ as r^2 becomes large.

Conclusions

It has been found that as the rise of an arch increases, the free vibration behavior in the fundamental mode first exhibits a softening trend due to curvature. This trend is reversed as the rise parameter exceeds the value given in Eq. (43). A neutral limit is ultimately approached for large values of rise.

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Reprinted from

Int. J. Solids Structures, 1973, Vol. 9, pp. 581 to 590. Pergamon Press. Printed in Great Britain

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PERGAMON PRESS

OXFORD NEW YORK LONDON PARIS

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INTRODUCTION

A THEORY of initial postbuckling behavior has been developed by Koiter [1, 2] which permits the first order effects of finite displacements and initial imperfections on the buckling process to be assessed. Although the original work was based upon potential energy considerations, Budiansky and Hutchinson [3] and Budiansky [4] rederived the essentials of the theory by another method which is based upon virtual work. In the present paper an approach is developed for the analysis of nonlinear free vibrations that is in much the same spirit. The approach is quite analogous to the treatment of Koiter's theory in Ref. [4].

A perturbation approach that is applicable to nonlinear partial differential equations which possess periodic solutions has been outlined by Keller [5] and applied by Keller and Ting [6]. The essential features of the perturbation approach are used in the present development. Solutions to the governing equations are sought as a power series in the amplitude of the linear vibration mode and higher order effects are systematically generated by successive perturbation equations.

OUTLINE OF THE THEORY

To facilitate a concise presentation of the theory, the functional notation used by Budiansky [4] will be employed. The motion of the structure produces generalized displacement $\mathbf{u}$, strain $\boldsymbol{\gamma}$ and stress $\boldsymbol{\sigma}$. The dynamics of the system is established by Hamilton's principle, which is symbolically written

$$\int_0^{2\pi/\omega} \left[\delta\left(\frac{1}{2} M\left(\frac{\partial \mathbf{u}}{\partial t}\right) \cdot \frac{\partial \mathbf{u}}{\partial t}\right) - \boldsymbol{\sigma} \cdot \delta\boldsymbol{\gamma} \right] \mathrm{d}t = 0. \tag{1}$$

The "dot" operation signifies the appropriate inner multiplication of variables and integration of the result over the entire structure. The generalized mass operator M is assumed to be homogeneous and linear with the property that

$$M(\mathbf{u}) . \mathbf{v} = M(\mathbf{v}) . \mathbf{u} \tag{2}$$

for all **u** and **v**. Since only periodic motion is to be considered, the limits on the integral over time correspond to a single period of the motion. ω is the circular frequency of the vibration such that

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{u}\left(\mathbf{r}, t+\frac{2\pi}{\omega}\right) \tag{3}$$

r is the position vector to an arbitrary point in the structure.

If a new time variable $\tau = \omega t$ is introduced in equation (1), this equation may be replaced by

$$\int_0^{2\pi} [\omega^2\delta(\tfrac{1}{2}M(\dot{\mathbf{u}}) . \dot{\mathbf{u}})-\boldsymbol{\sigma} . \delta\boldsymbol{\gamma}]\, d\tau = 0. \tag{4}$$

In the above, the notation $(\dot{\ }) = \partial(\ )/\partial\tau$ has been used. An integration by parts results in

$$\omega^2 M(\dot{\mathbf{u}}) . \delta\mathbf{u}|_0^{2\pi} - \int_0^{2\pi} [\omega^2 M(\ddot{\mathbf{u}}) . \delta\mathbf{u}+\boldsymbol{\sigma} . \delta\boldsymbol{\gamma}]\, d\tau = 0. \tag{5}$$

The boundary terms vanish by reason of periodicity and we are left with

$$\int_0^{2\pi} [\omega^2 M(\ddot{\mathbf{u}}) . \delta\mathbf{u}+\boldsymbol{\sigma} . \delta\boldsymbol{\gamma}]\, d\tau = 0 \tag{6}$$

$\delta\mathbf{u}$ is any virtual displacement that is consistent with all the kinematic boundary conditions imposed on the structure.

Equation (6) is supplemented by the strain–displacement relation

$$\boldsymbol{\gamma} = L_1(\mathbf{u})+\tfrac{1}{2}L_2(\mathbf{u}) \tag{7}$$

where L_1 and L_2 are homogeneous linear and quadratic functionals, respectively. In addition, the homogeneous bilinear functional L_{11} is defined by the following equation:

$$L_2(\mathbf{u}+\mathbf{v}) = L_2(\mathbf{u})+2L_{11}(\mathbf{u}, \mathbf{v})+L_2(\mathbf{v}). \tag{8}$$

It follows that $L_{11}(\mathbf{u}, \mathbf{v}) = L_{11}(\mathbf{v}, \mathbf{u})$ and $L_{11}(\mathbf{u}, \mathbf{u}) = L_2(\mathbf{u})$. If use is made of the above definitions, the variation of the generalized strain can be written as

$$\delta\boldsymbol{\gamma} = \delta\mathbf{e}+L_{11}(\mathbf{u}, \delta\mathbf{u}) \tag{9}$$

where

$$\mathbf{e} = L_1(\mathbf{u}) \tag{10}$$

is the linearized strain measure.

For Hookean (linear) elastic structures, the stress–strain relation can be written in the form

$$\boldsymbol{\sigma} = H(\boldsymbol{\gamma}) \tag{11}$$

where H is a homogeneous linear function. The following reciprocity relation

$$\boldsymbol{\sigma}^{(1)} . \boldsymbol{\gamma}^{(2)} = \boldsymbol{\sigma}^{(2)} . \boldsymbol{\gamma}^{(1)} \tag{12}$$

will be assumed also; "1" and "2" are any arbitrary states of stress and strain.

The vibration modes and frequencies of the linearized theory can be found by setting

$$\mathbf{u} = \xi\mathbf{u}_1 \qquad \boldsymbol{\gamma} = \xi\mathbf{e}_1 \qquad \boldsymbol{\sigma} = \xi\boldsymbol{\sigma}_1 \tag{13}$$

ξ is an amplitude parameter associated with the mode $\mathbf{u}_1$ which has natural frequency ω_0. If equation (13) is substituted into equation (6) and only linear terms are retained, we obtain

$$\int_0^{2\pi} [\omega_0^2 M(\ddot{\mathbf{u}}_1) . \delta\mathbf{u} + \boldsymbol{\sigma}_1 . \delta\mathbf{e}] \, d\tau = 0. \tag{14}$$

Equating the integrand to zero yields the linearized equation of motion.

If we now set $\delta\mathbf{u} = \mathbf{u}_1$ in the above equation, we obtain an expression for ω_0^2.

$$\begin{aligned} \omega_0^2 &= \frac{-\int_0^{2\pi} \boldsymbol{\sigma}_1 . \mathbf{e}_1 \, d\tau}{\int_0^{2\pi} M(\ddot{\mathbf{u}}_1) . \mathbf{u}_1 \, d\tau} \\ &= \frac{\int_0^{2\pi} \boldsymbol{\sigma}_1 . \mathbf{e}_1 \, d\tau}{\int_0^{2\pi} M(\dot{\mathbf{u}}_1) . \dot{\mathbf{u}}_1 \, d\tau}. \end{aligned} \tag{15}$$

This is analogous to a Rayleigh quotient for the natural frequency ω_0.

We assume at this point that a single mode $\mathbf{u}_1$ is associated with the natural frequency ω_0. The case of multiple modes corresponding to the same natural frequency will be discussed later.

To discover how the structure behaves for finite amplitudes, we assume

$$\begin{aligned} \mathbf{u} &= \xi\mathbf{u}_1 + \xi^2\mathbf{u}_2 + \ldots \\ \boldsymbol{\gamma} &= \xi\mathbf{e}_1 + \xi^2(\mathbf{e}_2 + \tfrac{1}{2}L_2(\mathbf{u}_1)) + \ldots \\ \boldsymbol{\sigma} &= \xi\boldsymbol{\sigma}_1 + \xi^2\boldsymbol{\sigma}_2 + \ldots \end{aligned} \tag{16}$$

where, in order to make the expansions unique, the displacement increments $\mathbf{u}_2, \mathbf{u}_3, \ldots$ are orthogonalized with respect to $\mathbf{u}_1$ in the sense that

$$M(\dot{\mathbf{u}}_1) . \dot{\mathbf{u}}_k = M(\ddot{\mathbf{u}}_1) . \mathbf{u}_k = 0 \qquad (k \neq 1). \tag{17}$$

This relation, together with equation (12), also implies that

$$\boldsymbol{\sigma}_1 . \mathbf{e}_k = 0 \qquad (k \neq 1) \tag{18}$$

which by virtue of the reciprocity relation (12) implies further that

$$H(\mathbf{e}_k) . \mathbf{e}_1 = 0 \qquad (k \neq 1). \tag{19}$$

The substitution of (16) into (6) yields

$$\begin{aligned} \int_0^{2\pi} \{ & \xi(\omega^2 M(\ddot{\mathbf{u}}_1) . \delta\mathbf{u} + \boldsymbol{\sigma}_1 . \delta\mathbf{e}) + \xi^2(\omega^2 M(\ddot{\mathbf{u}}_2) . \delta\mathbf{u} + \boldsymbol{\sigma}_2 . \delta\mathbf{e} + \boldsymbol{\sigma}_1 . L_{11}(\mathbf{u}_1, \delta\mathbf{u})) \\ & + \xi^3(\omega^2 M(\ddot{\mathbf{u}}_3) . \delta\mathbf{u} + \boldsymbol{\sigma}_3 . \delta\mathbf{e} + \boldsymbol{\sigma}_1 . L_{11}(\mathbf{u}_2, \delta\mathbf{u}) + \boldsymbol{\sigma}_2 . L_{11}(\mathbf{u}_1, \delta\mathbf{u})) + \ldots \} \, d\tau = 0. \end{aligned} \tag{20}$$

If we now set $\delta\mathbf{u} = \mathbf{u}_1$, $\delta\mathbf{e} = \mathbf{e}_1$ in this expression and introduce (15), the following result is obtained:

$$\begin{aligned} \int_0^{2\pi} \Bigg\{ & \xi\left(1 - \frac{\omega^2}{\omega_0^2}\right)\boldsymbol{\sigma}_1 . \mathbf{e}_1 + \xi^2(\boldsymbol{\sigma}_2 . \mathbf{e}_1 + \boldsymbol{\sigma}_1 . L_{11}(\mathbf{u}_1, \mathbf{u}_1)) \\ & + \xi^3(\boldsymbol{\sigma}_3 . \mathbf{e}_1 + \boldsymbol{\sigma}_1 . L_{11}(\mathbf{u}_1, \mathbf{u}_2) + \boldsymbol{\sigma}_2 . L_{11}(\mathbf{u}_1, \mathbf{u}_1)) + \ldots \Bigg\} \, d\tau = 0. \end{aligned}$$

However, the reciprocity relation (12) permits further simplification as

$$\sigma_2 \cdot e_1 = \sigma_1 \cdot \gamma_2 = \sigma_1 \cdot (e_2 + \tfrac{1}{2}L_{11}(u_1, u_1)) = \tfrac{1}{2}\sigma_1 \cdot L_{11}(u_1, u_1) \tag{21}$$

and

$$\sigma_3 \cdot e_1 = \sigma_1 \cdot \gamma_3 = \sigma_1 \cdot (e_3 + L_{11}(u_1, u_2)) = \sigma_1 \cdot L_{11}(u_1, u_2). \tag{22}$$

Consequently,

$$\int_0^{2\pi} \left\{ \xi\left(1 - \frac{\omega^2}{\omega_0^2}\right)\sigma_1 \cdot e_1 + \xi^2(\tfrac{3}{2}\sigma_1 \cdot L_{11}(u_1, u_1)) + \xi^3(2\sigma_1 \cdot L_{11}(u_1, u_2) + \sigma_2 \cdot L_{11}(u_1, u_1)) + \ldots \right\} d\tau = 0 \tag{23}$$

and we have the asymptotic relation

$$\frac{\omega^2}{\omega_0^2} = 1 + A\xi + B\xi^2 + \ldots \tag{24}$$

where

$$A = \frac{\int_0^{2\pi} \frac{3}{2}\sigma_1 \cdot L_{11}(u_1, u_1) d\tau}{\int_0^{2\pi} \sigma_1 \cdot e_1 \, d\tau} = \frac{\int_0^{2\pi} \frac{3}{2}\sigma_1 \cdot L_{11}(u_1, u_1) \, d\tau}{\omega_0^2 \int_0^{2\pi} M(\dot{u}_1) \cdot \dot{u}_1 \, d\tau} \tag{25}$$

and

$$B = \frac{\int_0^{2\pi} (2\sigma_1 \cdot L_{11}(u_1, u_2) + \sigma_2 \cdot L(u_1, u_1)) \, d\tau}{\omega_0^2 \int_0^{2\pi} M(\dot{u}_1) \cdot \dot{u}_1 \, d\tau}. \tag{26}$$

If A is nonzero, the structure can exhibit a softening characteristic with the frequency decreasing for finite amplitudes for $A\xi$ negative. If A is zero, a negative value of B corresponds to softening (decreasing frequency) and a positive, nonzero value corresponds to hardening (increasing frequency). If both A and B are zero, higher order terms must be investigated to discover the nature of finite amplitude effects.

In the evaluation of B a solution for u_2 and σ_2 is required. It must be found from the second order term in equation (20). The variational equation of motion is, therefore,

$$\omega^2 M(\ddot{u}_2) \cdot \delta u + \sigma_2 \cdot \delta e + \sigma_1 \cdot L_{11}(u_1, \delta u) = 0 \tag{27}$$

and

$$\gamma_2 = e_2 + \tfrac{1}{2}L(u_1, u_1) \qquad \sigma_2 = H(\gamma_2)$$

δu is orthogonal to u_1 in the sense of equation (17).

MULTIPLE MODE SITUATIONS

If more than one mode corresponds to the same natural frequency found from the linearized theory, the above described solution process requires modification. If we assume that k modes correspond to the same frequency, with the linearly independent modes being identified as $\mathbf{u}_{11}, \mathbf{u}_{12}, \ldots, \mathbf{u}_{1k}$, then the expansion for displacement in equation (16) must be replaced by

$$\mathbf{u} = \sum_{i=1}^{k} \xi_i \mathbf{u}_{1i} + \mathbf{w} \tag{28}$$

where the modes are orthogonalized with respect to each other and to the additional displacement $\mathbf{w}$. Also we write

$$\boldsymbol{\sigma} = \sum_{i=1}^{k} \xi_i \boldsymbol{\sigma}_{1i} + \mathbf{s}. \tag{29}$$

It is possible to develop k equations of the same type as (23) which are obtained by setting $\delta\mathbf{u}$ equal to $\mathbf{u}_{11}, \mathbf{u}_{12}, \ldots, \mathbf{u}_{1k}$. These equations, retaining only quadratic terms in the ξ_i's as was done by Budiansky and Hutchinson [3] and thus neglecting the effects of $\mathbf{w}$, must be solved simultaneously. Since the equations are homogeneous, only amplitude ratios can be found. A higher order analysis can be made, but the solution process is far more difficult than the case of a single vibration mode.

APPLICATION TO FLEXURAL VIBRATIONS OF BEAMS

Consider the uniform, simply supported beam shown in Fig. 1. An axial force is induced in the beam due to finite amplitude vibrations because the supports are assumed to be immovable. Let U and W be the longitudinal and transverse components of displacement, N be the axial force and X and Z be the coordinates shown in the figure. The average axial strain ε is given by

$$\varepsilon = U_{,X} + \tfrac{1}{2}(W_{,X})^2 = \frac{N}{EA}, \tag{30}$$

where E is Young's modulus and A is the cross-sectional area. Since the supports cannot move apart

$$U(L) - U(0) = \int_0^L U_{,X}\,\mathrm{d}X = 0 \tag{31}$$

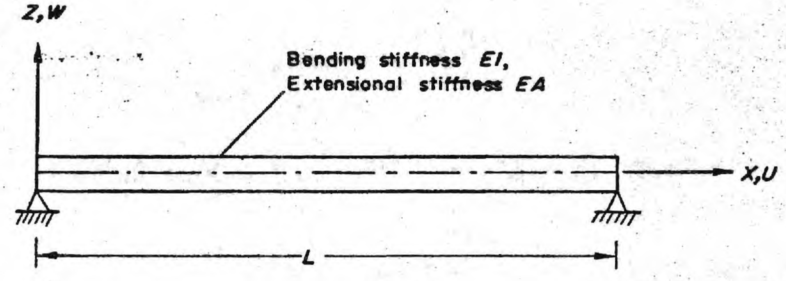

FIG. 1.

which implies

$$N = \frac{EA}{2L}\int_0^L (W_{,X})^2\,\mathrm{d}X. \tag{32}$$

The equation governing transverse vibrations of the beam is

$$mW_{,tt}+EIW_{,XXXX}-NW_{,XX} = 0, \tag{33}$$

where m is the mass per unit length of the beam and I is the second moment of area. If we introduce the following variables and parameters,

$$n = \frac{NL^2}{\pi^2 EI} \quad x = \frac{\pi X}{L} \quad w = \frac{W}{\sqrt{(\pi\rho)}} \quad \rho = \sqrt{\left(\frac{I}{A}\right)} \quad \omega_1^2 = \frac{\pi^4 EI}{mL^4} \tag{34}$$

then equations (32) and (33) can be written in a convenient dimensionless form. They become

$$n = \frac{1}{2}\int_0^\pi (w_{,x})^2\,\mathrm{d}x \tag{35}$$

$$\frac{1}{(\omega_1)^2}w_{,tt}+w_{,xxxx}-nw_{,xx} = 0. \tag{36}$$

Equation (36) is subject to the boundary conditions

$$\begin{aligned} w(0,t) &= w_{,xx}(0,t) = 0 \\ w(\pi,t) &= w_{,xx}(\pi,t) = 0. \end{aligned} \tag{37}$$

We now set $\tau = \omega t$, $\Omega = \omega^2/\omega_1^2$, and, in view of the structure of the equations,

$$\begin{aligned} w &= \xi w_1+\xi^3 w_3+\ldots \\ n &= \xi^2 n_2+\xi^4 n_4+\ldots. \end{aligned} \tag{38}$$

The first approximation involves only w_1. The linearized equation is

$$\Omega\ddot{w}_1+w_{1,xxxx} = 0. \tag{39}$$

The solution is of the form

$$w_1 = \cos\tau\sin kx, \tag{40}$$

where k is an integer. This solution yields

$$\Omega_0 = k^4 \tag{41}$$

for the dimensionless frequency parameter.

The second approximation is simply

$$n_2 = \frac{1}{2}\int_0^\pi (w_{1,x})^2\,\mathrm{d}x = \frac{\pi k^2}{8}(1+\cos 2\tau). \tag{42}$$

The coefficient A in the expansion (24) is zero. B is determined to be

$$B = \frac{\int_0^{2\pi}\int_0^\pi n_2(w_{1,x})^2\,\mathrm{d}x\,\mathrm{d}\tau}{\Omega_0\int_0^{2\pi}\int_0^\pi(\dot{w}_1)^2\,\mathrm{d}x\,\mathrm{d}\tau} = \frac{3\pi}{16}. \tag{43}$$

Consequently, we have the asymptotic relation

$$\frac{\Omega}{\Omega_0} = 1+\frac{3\pi}{16}\xi^2+\dots \tag{44}$$

This asymptotic approximation can be shown to be in complete agreement with an asymptotic representation of the solution obtained by Woinowsky–Krieger [7] in terms of elliptic functions and with the solutions obtained by Chu and Herrmann [8] using two different methods.

APPLICATION TO FLEXURAL VIBRATIONS OF RECTANGULAR PLATES

A rectangular plate of length a, width b and thickness h is shown in Fig. 2, along with notation and a sign convention. The origin of surface coordinates (X, Y) is taken to be the corner of the plate, and U, V and W are midsurface displacement components. The plate is assumed to be simply supported on all four edges and relative motions of the edges are assumed to be prevented. Under these conditions, membrane stresses will be induced in the plate due to transverse flexural vibrations of finite amplitude.

The analysis is based upon the following equations, which are equivalent to the dynamic version of von Karman's equations used by Chu and Herrmann [8]:

$$\nabla^4 f = (w_{,xy})^2 - w_{,xx}w_{,yy} \tag{45}$$

$$\frac{4}{(\omega_{sp})^2}w_{,tt}+\nabla^4 w - f_{,yy}w_{,xx}+2f_{,xy}w_{,xy}-f_{,xx}w_{,yy} = 0 \tag{46}$$

$$u_{,x} = f_{,yy}-\nu f_{,xx}-\tfrac{1}{2}(w_{,x})^2 \tag{47}$$

$$v_{,y} = f_{,xx}-\nu f_{,yy}-\tfrac{1}{2}(w_{,y})^2. \tag{48}$$

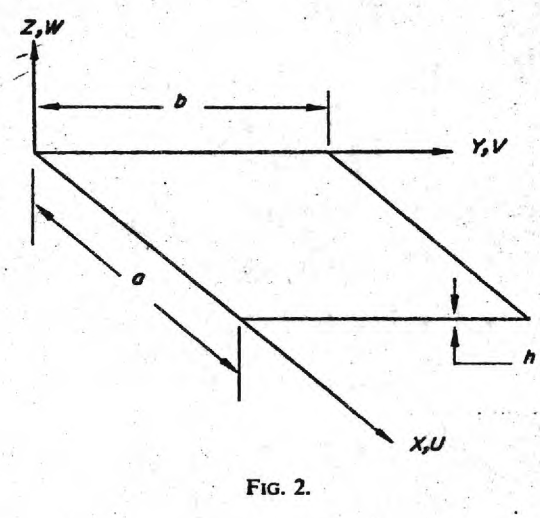

FIG. 2.

The dimensionless variables and parameters are listed below, along with any other important quantities.

$$x = \frac{\pi X}{b} \qquad y = \frac{\pi Y}{b} \qquad w = \frac{[12(1-\nu^2)]^{\frac{1}{2}} W}{h} \qquad u = \frac{4\pi E}{\sigma_p b} U \qquad v = \frac{4\pi E}{\sigma_p b} V$$

$$(\omega_{sp})^2 = \frac{Eh^3}{3(1-\nu^2)m}\left(\frac{\pi}{b}\right)^4 \qquad \mu = \frac{b}{a} \qquad \sigma_p = \frac{E}{3(1-\nu^2)}\left(\frac{\pi h}{b}\right)^2$$

$$\nabla^2(\ ) = \frac{\partial^2(\ )}{\partial x^2} + \frac{\partial^2(\ )}{\partial y^2} \qquad \nabla^4(\ ) = \nabla^2[\nabla^2(\ )]. \tag{49}$$

The stress parameter σ_p is the compressive buckling stress of long or square simply supported plates; ω_{sp} is the fundamental circular frequency of a square plate. ν is Poisson's ratio and f is a dimensionless Airy stress function defined such that the membrane stresses, σ_x^0, σ_y^0, and τ_{xy}^0, are given by the relation

$$\{\sigma_x^0, \sigma_y^0, \tau_{xy}^0\} = \frac{\sigma_p}{4}\{f_{,yy}, f_{,xx}, -f_{,xy}\}. \tag{50}$$

The above partial differential equations are subject to the following boundary conditions:

$$w(0, y) = w\left(\frac{\pi}{\mu}, y\right) = w(x, 0) = w(x, \pi) = 0 \tag{51}$$

$$w_{,xx}(0, y) = w_{,xx}\left(\frac{\pi}{\mu}, y\right) = w_{,yy}(x, 0) = w_{,yy}(x, \pi) = 0 \tag{52}$$

$$v(x, 0) = v(x, \pi) = 0 \tag{53}$$

$$u(0, y) = u\left(\frac{\pi}{\mu}, y\right) = 0. \tag{54}$$

$\mu = b/a$ is the aspect ratio of the plate.

We set $\tau = \omega t$, $\Omega = \dfrac{\omega^2}{(\omega_{sp})^2}$ and

$$\begin{aligned} w &= \xi w_1 + \xi^3 w_3 + \ldots \\ f &= \xi^2 f_2 + \xi^4 f_4 + \ldots. \end{aligned} \tag{55}$$

The first order equation involves only w_1 and is

$$4\Omega \ddot{w}_1 + \nabla^4 w_1 = 0. \tag{56}$$

We take the solution corresponding to the fundamental mode of the plate in the form

$$w_1 = \sin \mu x \sin y \sin \tau, \tag{57}$$

which leads to the linearized frequency ratio

$$\Omega_0 = \frac{(\mu^2+1)^2}{4}. \tag{58}$$

The second order equation is

$$\nabla^4 f_2 = (w_{1,xy})^2 - w_{1,xx}w_{1,yy} = \frac{\mu^2(1-\cos 2\tau)}{4}(\cos 2\mu x + \cos 2y). \tag{59}$$

We take the following solution for f_2:

$$f_2 = \tfrac{1}{64}(1-\cos 2\tau)\left[\frac{1}{\mu^2}\cos 2\mu x + \mu^2 \cos 2y + 2\alpha x^2 + 2\beta y^2\right] \tag{60}$$

α and β are constants which must be evaluated so as to satisfy the boundary conditions (53) and (54). The tangential displacement parameters can be determined with f_2 and w_1 known; they are

$$u_2 = \frac{1}{32\mu}(1-\cos 2\tau)\sin 2\mu x(\nu - \mu^2 + \mu^2 \cos 2y) \tag{61}$$

$$v_2 = \tfrac{1}{32}(1-\cos 2\tau)\sin 2y(\nu\mu^2 - 1 + \cos 2\mu x). \tag{62}$$

It can be verified that these expressions satisfy (53) and (54) and that we must require that

$$\alpha = \frac{1+\nu\mu^2}{(1-\nu^2)} \tag{63}$$

and

$$\beta = \frac{(\mu^2+\nu)}{(1-\nu^2)}. \tag{64}$$

Again, as for the beam, the coefficient A in the expansion (24) is zero. For the plate, B is found to be

$$B = \frac{\int_0^{2\pi}\int_0^{\pi/\mu}\int_0^{\pi}[f_{2,yy}(w_{1,x})^2 + f_{2,xx}(w_{1,y})^2]\,dx\,dy\,d\tau}{4\Omega_0\int_0^{2\pi}\int_0^{\pi/\mu}\int_0^{\pi}(\dot{w})^2\,dx\,dy\,d\tau} \tag{65}$$

$$= \frac{3}{32(\mu^2+1)}\left[\frac{\mu^2+2\nu\mu^2+1}{(1-\nu^2)} + \frac{1}{2}(\mu^4+1)\right].$$

Consequently, we have the asymptotic relation

$$\frac{\Omega}{\Omega_0} \cong 1 + \frac{3}{32(\mu^2+1)}\left[\frac{\mu^2+2\nu\mu^2+1}{(1-\nu^2)} + \frac{1}{2}(\mu^4+1)\right]\xi^2. \tag{66}$$

A solution to this problem has been obtained by Chu and Herrmann [8] in terms of elliptic functions. A careful study of their solution and an asymptotic representation of it for small amplitudes shows that there is complete agreement between it and the present solution, albeit a lengthy process.

CONCLUDING REMARKS

A general approach to nonlinear vibrations of elastic structures has been developed which provides information regarding the first order effects of finite displacements. It relies upon the use of Hamilton's principle and a perturbation procedure to obtain analytical results. The theory effectively reduces a nonlinear free vibration problem to a sequence of linear problems, only the first two of which usually need be solved to obtain an initial estimate of finite amplitude effects. The theory has been applied to beams and rectangular plates, structures for which solutions already exist, in order to illustrate the theory and demonstrate its usefulness.

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(*Received* 5 *May* 1972)

Абстракт—С помощью принципа Гамильтона и метода возмущений, выводится подход к решению нелинейных, свободных колебаний упругих конструкций. Теория аналогична к теории начального закритического поведения в смысле Койтера. Дает она информацию, касающуюся эффектов первого рода для конечных перемещений на частоту, период и динамические напряжения, возникающие во время свободного, незатухающего колебания конструкций. Ограничивается внимание к линейно-упругим конструкциям. Для иллюстрации, теория применяется к задаче свободного колебания балок и прямоугольных пластинок.